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Algebraically explicit analytical solutions of unsteady 3-D nonlinear non-Fourier (hyperbolic) heat conduction

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Abstract

Analytical solutions of different nonlinear heat conduction equations are meaningful in heat transfer theory. In addition, they are very useful to computational heat transfer to verify numerical solutions and to develop numerical schemes, grid generation methods and so forth. However, none or very few explicit analytical solutions are known for nonlinear (thermal properties are functions of temperature) non-Fourier heat conduction equations. In this paper, some algebraically explicit analytical solutions of unsteady geometrical 3-D nonlinear non-Fourier heat conduction equation are derived. Some mathematical tools needed to extract such explicit exact solutions for complicated nonlinear partial differential equations are also discussed. For example, the little known and rarely used method of separating variables with addition is developed; matching relations between the functions of the thermal conductivity, of the freepath and of the volumetric specific heat are suggested. The main aim of this paper is to obtain some possibly explicit analytical solutions of the nonlinear and non-Fourier heat conduction equation as the benchmark solutions of computational heat transfer but not a specified solution for given initial and boundary conditions, therefore, the initial and boundary conditions are indeterminate before derivation and deduced from the solutions afterwards.

Keywords: Analytical solution; Heat conduction; Non-Fourier; Nonlinear; Separating variables with addition

Analytical solutions play a very important role in heat transfer. Many analytical solutions of constant coefficient equations played a key role in the early development of heat conduction. However, for variable thermal conductivity, free path and volumetric specific heat, the governing equations of non-Fourier heat conduction are nonlinear, and therefore it is difficult to find their analytical solutions. To the authors' knowledge, very few or even no explicit analytical solutions of unsteady nonlinear non-Fourier heat conduction can be found in the open literature at present. In order to fill some of the gaps in the field of unsteady nonlinear and non-Fourier heat conduction, it is meaningful to extract some analytical solutions. Therefore, some algebraically explicit analytical solutions of unsteady nonlinear compressible flow and heat conduction were recently given by one of the authors [1,2] to develop the theory of aerodynamics and heat conduction.

Moreover, besides theoretical meaning, analytical solutions can also be applied to check the accuracy, convergence and effectiveness of various numerical computation methods and their differencing schemes, grid generation ways and so on. The analytical solutions are therefore very useful even for the newly rapidly developing computational heat conduction. For example, in the fluid dynamics field, several analytical solutions that simulate the 3-D potential flow in turbomachine cascades were given by one of the authors [3]; these solutions have been used successfully by independent researchers to verify the accuracy and exactness of their computational methods [4–6].

Several algebraically explicit analytical solutions of unsteady nonlinear non-Fourier heat conduction are derived in this paper, both to develop the theory and to serve as benchmark solutions for numerical calculations.

1. Governing equation and solution method

When the thermal conductivity, free path and volumetric specific heat are functions of temperature, the governing equa-

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Nomenclature

C_1–C_{11}	arbitrary constants	
C	specific heat	$\mathrm{J\,kg^{-1}\,K^{-1}}$
k	arbitrary constant	
K	thermal conductivity	$\mathrm{W\,m^{-1}\,K^{-1}}$
l	free path	m
m, n	arbitrary constants	
t	time coordinate	s
x, y, z	geometric coordinates	m
Greek symbols		
ρ	density	$\mathrm{kg\,m^{-3}}$
θ	temperature	K
τ	thermal relaxation time	s

tion of unsteady 3-D nonlinear non-Fourier heat conduction can be expressed as follows

$$\rho C\left(\frac{\partial\theta}{\partial t}+\tau\frac{\partial^2\theta}{\partial t^2}\right) = \frac{\partial}{\partial x}\left(K\frac{\partial\theta}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial\theta}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial\theta}{\partial z}\right) \tag{1}$$

where ρ, C, θ, τ and K are density, specific heat, temperature, thermal relaxation time and thermal conductivity; t, x, y and z are time and geometric coordinates. Volumetric specific heat is defined as ρC. According to Ref. [7,8],

$$\tau=\rho Cl^2/3K \tag{2}$$

where l is free path.

It is emphasized that the main aim of this paper is to obtain some possibly explicit analytical solutions of Eq. (1) but not a specified solution for given initial and boundary conditions: therefore, the initial and boundary conditions are indeterminate before derivation and deduced from the solutions afterwards. As it will be shown below, this makes the derivation procedure easier. In order to derive explicit analytical solutions, another important point is that the function of thermal conductivity, the function of free path and the function of volumetric specific heat have to be matchable in some degree. In some cases the method of separating variables with addition [1,2,9–18] is applied to solve the equation: in this method, it is assumed that the unknown solution $\theta=\theta(t,x,y,z)=T(t)+X(x)+Y(y)+Z(z)$ replaces $\theta=(t,x,y,z)=T(t)\cdot X(x)\cdot Y(y)\cdot Z(z)$ in the common method of separating variables.

Actually, all solutions given in this paper can be proven easily by substituting them into the governing equation.

2. Solutions with the method of separating variables with addition

As mentioned in previous paragraph, the existence of some relation between the thermal properties is one of the requirements to obtain explicit analytical solutions of nonlinear equations. In our treatment, the following functions of temperature are assumed:

$$\rho C=me^{n\theta} \tag{3}$$

$$K=ke^{n\theta} \tag{4}$$

When $n=0$, Eqs. (3) and (4) represent constant ρC and K, when $n>0$ or <0, ρC and K increase or decrease with temperature. Commonly the absolute value of n is not high. It is assumed $l=\text{const}$ as well, then the thermal relaxation time τ is a constant also and equal to $ml^2/(3k)$ according to Eqs. (2)–(4). With such relations and the method of separating variables with addition, Eq. (1) can be simplified as following

$$m\left[T'+ml^2T''/(3k)\right] = k\left(X''+nX'^2+Y''+nY'^2+Z''+nZ'^2\right) \tag{5}$$

Eq. (5) can be separated into following equation

$$T'+ml^2T''/(3k) = \pm C_1^2 = k/m\left(X''+nX'^2+Y''+nY'^2+Z''+nZ'^2\right) \tag{6}$$

and then it is deduced from left-hand side of Eq. (6)

$$T''+3kT'/\left(ml^2\right)\mp 3kC_1^2/\left(ml^2\right)=0 \tag{7}$$

The right-hand side of Eq. (6) can be separated again as follows

$$X''+nX'^2=C_5=\pm mC_1^2/k-Y''-nY'^2-Z''-nZ'^2 \tag{8}$$

and

$$Y''+nY'^2=C_7=\pm mC_1^2/k-C_5-Z''-nZ'^2 \tag{9}$$

Therefore, we can derive the solution of governing equation by solving Eq. (7), the left-hand side of Eq. (8) and both sides of Eq. (9). The final solution is $T+X+Y+Z$.

The ordinary differential equation (7) is easy to solve, its solution is

$$T=\pm C_1^2t+C_2\exp(-t/\tau)+C_4 \tag{10}$$

For left-hand side of Eq. (8), the solution is different for different values of n and C_5. When they are all positive or negative, the solution is

$$X=\ln\left\{\left[1-\exp\left[2\sqrt{C_5n}(C_6-x)\right]\right]^2 / \exp\left[2\sqrt{C_5n}(C_6-x)\right]\right\}/(2n) \tag{11}$$

Similarly, for left-hand side of Eq. (9), when n and C_7 are all positive or negative, the solution is

$$Y=\ln\left\{\left[1-\exp\left[2\sqrt{C_7n}(C_8-y)\right]\right]^2 / \exp\left[2\sqrt{C_7n}(C_8-y)\right]\right\}/(2n) \tag{12}$$

for right-hand side of Eq. (9), when n and $\pm mC_1^2/k-C_5-C_7$ are all positive or negative, the solution is

$$Z=\ln\left\{\left[1-\exp\left[2\sqrt{n\left(\pm mC_1^2/k-C_5-C_7\right)}(C_9-z)\right]\right]^2 / \exp\left[2\sqrt{n\left(\pm mC_1^2/k-C_5-C_7\right)}(C_9-z)\right]\right\}/(2n) \tag{13}$$

Then a solution of Eq. (1) can be expressed by the summation of Eqs. (10)–(13). The initial and boundary conditions of this solution can be obtained by substituting some given initial time and boundary coordinates into the solution. For example, if $t = 0$, the initial condition is

$$\theta = C_2 + C_4 + X + Y + Z \tag{14}$$

For the solutions discussed in the following paragraphs, the "decision" about the proper initial and boundary conditions will be made the same way.

Since τ must be positive, it can be understood from Eq. (10) that when $C_1 = 0$ and $C_2 \neq 0$ the temperature field of this solution will be a decreasing function of time, converging for $t \to \infty$ to a steady temperature distribution $C_4 + X + Y + Z$. The transient will be basically completed in a time interval of the same order of magnitude as the thermal relaxation time. This is a common law in non-Fourier processes.

When $C_1 \neq 0$ and $C_2 = 0$, this solution represents a temperature field with linear temperature increase or decrease.

The solution Eqs. (10)–(13) is a mathematical 4-D solution that we believe is new, and has its theoretical significance. Furthermore, it may be used as a non-trivial benchmark for computational heat transfer calculations.

Since n cannot be zero in Eqs. (11)–(13), we cannot simply degenerate them to get a constant thermal property solution.

The constant thermal property solution with $\rho C = m$ and $K = k$ ($n = 0$) can be deduced similarly, the final expression is as follows

$$\begin{aligned}\theta = {} & \pm C_1 t^2 + C_2 \exp(-t/\tau) + C_5 x^2/2 + C_6 x + C_7 y/2 + C_8 y \\ & + \left(\pm C_1^2 m/k - C_5 - C_7\right) z^2/2 + C_9 z + C_4\end{aligned} \tag{15}$$

The temperature variation with time is the same as Eq. (10), but the temperature variation with geometric coordinates is parabolic.

If n and C_5 (as well as $C_7, mC_1^2/k - C_5 - C_7$) have different sign, the solution of Eq. (7) is the same as previous Eq. (10), but the solutions of Eqs. (8) and (9) are different, and can be expressed as follows

$$X = \ln\left\{\cos\left\lfloor n\sqrt{-C_5/n}(C_6 - x)\right\rfloor\right\}/n \tag{16}$$

$$Y = \ln\left\{\cos\left\lfloor n\sqrt{-C_7/n}(C_8 - y)\right\rfloor\right\}/n \tag{17}$$

$$Z = \ln\left\{\cos\left\lfloor n\sqrt{\mp mC_1^2/k + C_5 + C_7}(C_9 - z)\right\rfloor\right\}/n \tag{18}$$

and the temperature distribution is the sum of Eqs. (16)–(18) and Eq. (10).

The discussion presented for the previous solutions applies here as well, and applies to the selection of initial and boundary conditions, the influence of the value of C_1 and C_2, the meaning of the individual solutions, etc.

3. Solutions with ordinary method of separating variables

Applying the ordinary method of separating variables to the governing equation (1), the matching relations between the thermal properties would be

$$\rho C = m\theta^n \tag{19}$$

and

$$K = k\theta^n \tag{20}$$

Similar to previous paragraph, if we assumed that free path $l = \text{const}$, then the thermal relaxation time τ is equal to $ml^2/(3k)$ and a constant also.

Solving Eqs. (19) and (20) by the ordinary method of separation of variables: $\theta = T \cdot X \cdot Y \cdot Z$, the following ordinary differential equations can be deduced

$$T'' + 3kT'/\left(ml^2\right) = \pm 3kC_1^2 T/\left(ml^2\right) \tag{21}$$

$$X'' + nX'^2/X - C_4 X = 0 \tag{22}$$

$$Y'' + nY'^2/Y - C_7 Y = 0 \tag{23}$$

$$Z'' + nZ'^2/Z - \left(\pm mC_1^2/k - C_4 - C_7\right)Z = 0 \tag{24}$$

If a positive sign is chosen in the right-hand side of Eq. (21), the solution is easily found to be:

$$T = C_2 \exp(r_1 t) + C_3 \exp(r_2 t) \tag{25}$$

where

$$r_{1,2} = -1/(2\tau) \mp \sqrt{1/\tau^2 + 4C_1^2/\tau}/2$$

When $C_3 = 0$, there is a limit process which approaches a steady temperature field, and the transient time scale is basically determined by the thermal relaxation time τ.

If the negative sign is chosen and $4C_1^2 < 1/\tau$ or $\tau < 1/(2C_1)^2$ the solution is still Eq. (25); but with

$$r_{1,2} = -1/(2\tau) \mp \sqrt{1/\tau^2 - 4C_1^2/\tau}/2$$

When $\tau = 1/(2C_1)^2$, the solution is

$$T = (C_2 + C_3 t)\exp\left[-t/(2\tau)\right] \tag{26}$$

When $4C_1^2 > 1/\tau$ or $\tau > 1/(2C_1)^2$, the solution is

$$\begin{aligned}T = {} & \exp\left[-t/(2\tau)\right]\left\lfloor C_2 \cos\left(\sqrt{4C_1^2/\tau - 1/\tau^2}\,t/2\right)\right. \\ & \left. + C_3 \sin\left(\sqrt{4C_1^2/\tau - 1/\tau^2}\,t/2\right)\right\rfloor\end{aligned} \tag{27}$$

The Eq. (27) represents a more general phenomenon in non-Fourier process: the attenuating approach process "carries" a heat wave.

By the way, the Fourier heat conduction means $\tau = 0$, and one of the results of this solution is that only when τ is larger than a certain value, the heat wave exists.

Algebraically explicit analytical solutions of ordinary differential Eqs. (22)–(24) can only be derived with some special value of n. Following solutions are given for $n = -3/2$. Of course, this absolute value of n seems too high for common thermal properties. However, the theoretical significance of the solution is still apparent.

When $C_4 > 0$, the solution of Eq. (22) is

$$X = C_6 \sec^2\left\lfloor\sqrt{C_4/2}(x + C_5)\right\rfloor \tag{28}$$

When $C_4 < 0$, the solution is

$$\begin{aligned}X = {} & \pm 8C_4 \exp(\pm\sqrt{-2C_4}x) \\ & /\left[C_5 \exp(\pm\sqrt{-2C_4}x) + C_6\right]^2\end{aligned} \tag{29}$$

Similarly, Eqs. (23) and (24) have the following solutions:

$$Y = C_9 \sec^2\left\lfloor \sqrt{C_7/2}(y + C_8)\right\rfloor \quad (C_7 > 0) \tag{30}$$

or

$$Y = \pm 8C_7 \exp(\pm\sqrt{-2C_7}y)/\left[C_8 \exp(\pm\sqrt{-2C_7}y) + C_9\right]^2 \quad (C_7 < 0) \tag{31}$$

and

$$Z = C_{11} \sec^2\left\lfloor \sqrt{\left(\pm mC_1^2/k - C_4 - C_7\right)/2}(z + C_{10})\right\rfloor \quad \left(\pm mC_1^2/k - C_4 - C_7 > 0\right) \tag{32}$$

or

$$Z = \pm 8\left(\pm mC_1^2/k - C_4 - C_7\right) \times \exp\left[\pm\sqrt{2\left(C_4 + C_7 \mp mC_1^2/k\right)}z\right] / \left\{C_{10} \exp\left[\pm\sqrt{2\left(C_4 + C_7 \mp mC_1^2/k\right)}z\right] + C_{11}\right\}^2 \quad \left(\pm mC_1^2/k - C_4 - C_7 < 0\right) \tag{33}$$

If the temperature in Eqs. (3) and (4) and Eqs. (19) and (20) is absolute temperature and has to be positive, then the product of T, X, Y and Z must remain positive as well.

In addition, there are some particular solutions originated by specific values of some constants. For example, when $C_1 = 0$:

$$T = C_2 \exp(-t/\tau) + C_3 \tag{34}$$

Which means that the temperature field will tend to a steady value C_3XYZ.

When $C_4 = 0$, or $C_7 = 0$, or $\pm mC_1^2/k = C_4 + C_7$, then

$$X = [C_5x + C_6]^{1/(n+1)} \tag{35}$$

or

$$Y = [C_8y + C_9]^{1/(n+1)} \tag{36}$$

or

$$Z = [C_{10}z + C_{11}]^{1/(n+1)} \tag{37}$$

Which means that the temperature distribution along geometric coordinates is an exponential function.

As mentioned before, the final solution is $TXYZ$. It has to be emphasized again that both Eqs. (28)–(33) and their final solutions are valid for $n = -3/2$ only. It does not influence the theory meaning and the role of benchmark solution of computational heat transfer.

It has to be remarked that, on the contrary, Eqs. (25)–(27) and (34)–(37) are valid for any values of n.

4. Conclusions

The non-Fourier heat conduction is a new branch of heat transfer of interest for many new high technology fields. Some algebraically explicit analytical solutions are derived for its nonlinear cases (thermal properties are functions of temperature) with two methods of separation of valuables and matchable relation of thermal properties. These results are both theoretically important and helpful in computational heat transfer as benchmark solutions.

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